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Self-dual fractal models for the low-field Hall effect near percolation threshold

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Abstract. Fractal lattices with the 'self-duality' property are proposed to imitate the geometric texture of two-dimensional percolating networks at percolation threshold. The exponents describing the power law dependence on the scale length of the Hall and Ohmic conductivities are found. The exponent for the Hall conductivity agrees with twice that for the Ohmic conductivity. The model is extended to describe the approach towards the threshold. It is found that the model shows the typical percolation behaviour as a function of a parameter p (the bond concentration).

1. Introduction

Recently, there has been increasing interest in exact mathematical fractals (Mandelbrot 1982, Vicsek 1983, Given and Mandelbrot 1983, Ben-Avraham and Havlin 1983, Blumenfeld and Aharony 1985, Martin and Keefer 1985). The main reason is that solution of many important equations of physics on these lattices adds to our understanding of the goemetric and topological properties that are relevant to modelling the corresponding physical processes. The percolating infinite cluster is one of the most intensively studied random fractals (Deutscher *et al* 1983, Stauffer 1979, 1985, Stanley and Coniglio 1983, Kirkpatrick 1979, Kapitulnik and Deutscher 1984). Various geometrical models have been proposed to imitate the infinite incipient cluster at the percolation threshold and it is of great interest in understanding the effects of these different geometries on the transport properties near the percolation threshold. Mandelbrot (1984a, b) and Mandelbrot and Given (1984) have presented fractal models for percolation clusters at criticality. The Mandelbrot models possess the geometric and topological properties very close to the infinite cluster at the percolation threshold. Nagatani (1985) has proposed a regular model to describe the approach towards the threshold p_c. The regular model is possessed of such a property that is self-similar (fractal) on smaller length scales than the connectedness length but becomes homogeneous on large length scales.

In this paper we try to imitate the critical behaviour of the Hall effect just above the percolation threshold with the help of regular fractals.

In order to determine the critical behaviour of the Hall conductivity near the percolation threshold of an isotropic composite, Bergman (1983) and Bergman *et al* (1983) realised the Hall problem on a two-component discrete lattice as follows: each element of the lattice is a doublet of identical conductors with an Ohmic conductance

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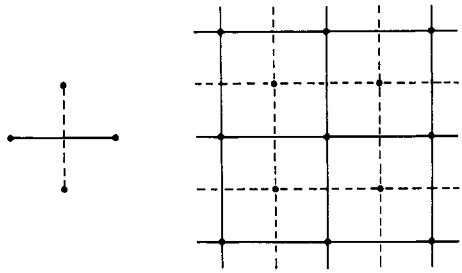

(a) (b)

Figure 1. Schematic drawing of a portion of the random-bond resistor networks used to realise the Hall effect in a discrete system in two dimensions. (a) A doublet of identical conductors with Ohmic and Hall conductances. A doublet is each element of the lattice. (b) A square-centred lattice of identical but unconnected mutually perpendicular doublets. The 2D network is composed of two unconnected (but correlated) simple-square resistor networks (shown by the full and broken lines respectively). The two networks are electrically unconnected in the absence of a magnetic field, but in the presence of a magnetic field these are correlated.

σ_1 or σ_2 that lies along the coordinate axes, and which are electrically unconnected in the absence of a magnetic field H (see figure 1). In the presence of an H field taken to lie along the z axis, a Hall current will flow through a conductor in the x direction that depends on its Hall conductance (λ_1 or λ_2) and on the voltage across the y conductor of the same doublet. The two types of doublets are placed randomly at all the sites of a square-centred lattice, and electrical connections are made only at the cell-edge centres. The random-bond resistor networks are electrically unconnected for $H=0$ but are correlated with each other by virtue of the doublets for a magnetic field along the z axis. The resulting doublet element, when used in constructing a random square-centred array, is the only way of obtaining a random array that is self-dual. The exact results on two-dimensional systems require this self-duality in order to be valid.

In order to imitate an infinite cluster in this lattice model of the Hall effect, we construct regular fractal models with the 'self-duality' property. These models consist of the branching Koch curves of which the fractal dimensionalities of the infinite cluster and its backbone agree with those of the Mandelbrot–Koch curve (Mandelbrot and Given 1984). The model is extended to describe the approach by making use of a rule of bond deletions on the square-centred lattice (Nagatani 1985). We produce simple objects which show the typical percolation behaviour as a function of a parameter p (the bond concentration).

2. Constructions of self-dual fractal lattices

We begin with the Mandelbrot–Koch curve proposed as the non-random model for the infinite cluster at criticality. Its generator is shown in figure 2(a). The fractal dimensionalities of the infinite cluster and its backbone are $\log_3 8(\sim 1.893)$ and $\log_3 6(\sim 1.631)$, respectively, very close to the known percolation values (Mandelbrot and Given 1984, Stauffer 1985). The dual lattice of this model is however not coincident with itself. We construct a regular fractal lattice with the 'self-duality' property and

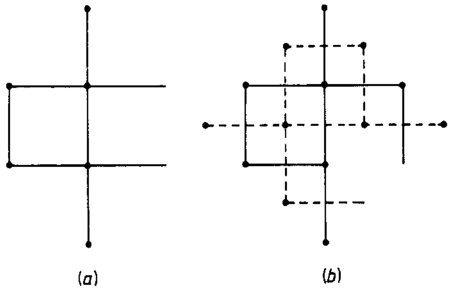

Figure 2. Generators of the fractals for the infinite cluster. (a) The Mandelbrot-Koch curve. (b) The branching Koch curve with the self-duality property (shown by the full lines). The dual lattice is shown by the broken lines. The two fractals shown by (a) and (b) have the same fractal dimensionality and the fractal dimensions of those backbones also coincide with each other.

the same fractal dimensionalities as the Mandelbrot-Koch curve. The generator of the self-dual fractal model is indicated by the full line in figure 2(b). The dual is represented by the broken line. The second stage of construction of the branching Koch curve, drawn with the use of a square initiator ABCD and the generator in figure 2(b), is shown in figure 3. This dual lattice is represented by the broken lines. The two lattices, being self-dual with each other and indicated by the full and broken lines, are electrically unconnected (for $H = 0$) but are correlated with each other by virtue of the unconnected doublets in the present of a magnetic field H (see figure 1).

One can construct other self-dual fractal lattices with a different scale factor. The generator of the fractal lattice with the scale factor $b = 5$ is shown in figure 4. The fractal dimensionalities of the infinite cluster and its backbone are $\log_5 21(\sim 1.892)$ and $\log_5 14(\sim 1.640)$ respectively, very close to the known percolation values.

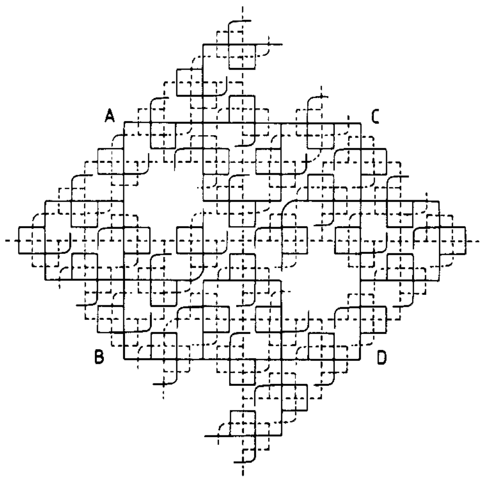

Figure 3. Second stage of construction of the Koch curve drawn with the use of a square initiator ABCD and the generator in figure 2(b). This dual lattice is represented by the broken lines. The two lattices indicated by the full and broken lines are electrically unconnected for $H = 0$ but are correlated with each other in the presence of a magnetic field H.

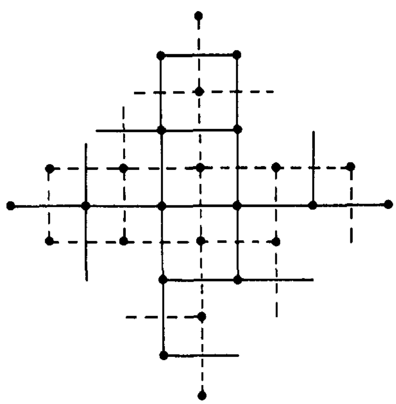

Figure 4. The generator of the fractal lattice with the scale factor $b = 5$. The self-dual lattice is shown by the broken lines. The fractal dimensions of the infinite cluster and its backbone are $D = 1.892\ldots$ and $D_b = 1.640\ldots$ respectively.

3. Renormalisation of conductivities

Consider now the Koch curve in figure 2(*b*). We must first calculate the total Ohmic and Hall conductivities between the endpoints in the x direction (or in the y direction), and then derive the exponent describing the power law dependence on scale length L of the conductivities.

In addition to the Ohmic conductance σ_a of each member a of the unit element, there is also a Hall conductance λ_a and Hall coefficient R_a, connected by

$$\lambda_a = \begin{cases} \sigma_a^2 R_a H & \text{for } a \perp H \\ 0 & \text{for } a \parallel H. \end{cases} \tag{1}$$

The current J_a is given by

$$J_a = \sigma_a V_a - \lambda_a V_{a\times H} \tag{2}$$

where $V_{a\times H}$ is the voltage across another conductor of the same unit element—the one that is perpendicular to both a and H. Current conservation at the internal point i leads to the following equation for the potentials V_j:

$$\sum_j \sigma_{ij}(V_i - V_j) + \sum_{ij\times H} \lambda_{ij} V_{ij\times H} = 0 \tag{3}$$

where the first summation over j indicates the sum over the nearest-neighbour sites to i, and the second summation over $ij \times H$ represents the sum over another conductor of the same unit element as the ij bond.

The self-similarity of the lattice on length scales leads to a natural decimation procedure. Two parameters are involved in the decimation procedure because of different types of conductivities. The idea involves eliminating the lowest scale potentials in equation (3). This procedure leads to a reduced set of equations describing the same physics on a lattice scaled down by a factor $b = 3$. This exact renormalisation leads to two renormalised conductivities. The decimation technique is schematically indicated in figure 5. The nodes represented by crosses on the left-hand lattice are eliminated, producing the renormalised right-hand lattice. The current, flowing through the renormalised bond in the x direction (or in the y direction), is given by

$$J_x = \tfrac{4}{11}\sigma(V_B - V_A) - (\tfrac{4}{11})^2\lambda(V_C - V_D) + O(\lambda^2) \tag{4}$$

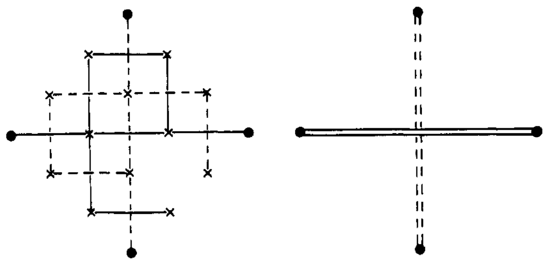

Figure 5. Schematic representation of the decimation technique for a part of the self-dual fractal lattices. The sites denoted by crosses on the left-hand lattices are eliminated, producing the right-hand renormalised doublet.

or

$$J_y = \tfrac{4}{11}\sigma(V_C - V_D) + (\tfrac{4}{11})^2\lambda(V_B - V_A) + O(\lambda^2)$$

where we omit the terms of a magnetic field H higher than the first-order term.

The renormalised Ohmic and low-field Hall conductivities are then given by

$$\sigma' = \tfrac{4}{11}\sigma \tag{5}$$

and

$$\lambda' = (\tfrac{4}{11})^2\lambda. \tag{6}$$

The exponents (t/ν and τ/ν), describing the power law dependence on scale length L of the Ohmic and Hall conductivities ($L^{-t/\nu}$ and $L^{-\tau/\nu}$), are given by

$$t/\nu = -\log(\sigma'/\sigma)/\log b = \log \tfrac{11}{4}/\log 3(\sim 0.9207) \tag{7}$$

and

$$\tau/\nu = -\log(\lambda'/\lambda)/\log b = 2\log \tfrac{11}{4}/\log 3 = 2(t/\nu). \tag{8}$$

It is found that the exponent for the low-field Hall conductivity agrees with being twice that for the Ohmic conductivity.

Similarly, for the fractal lattice shown by figure 4, we obtain the exponents of the conductivities

$$t/\nu = \log \tfrac{226}{49}/\log 5(\sim 0.9498) \tag{9}$$

and

$$\tau/\nu = 2(t/\nu). \tag{10}$$

4. The approach towards the threshold

We extend the fractal model for the infinite cluster at the criticality to describe the approach towards the threshold. We produce simple objects which show the typical percolation behaviour as a function of a parameter p (the bond concentration).

In general, every lattice bond has three choices in the bond percolation: it can be empty, with probability $1-p$; it can be part of the infinite network of occupied bonds, with probability pP_∞ (the percolation probability); or it can be part of one of the many finite clusters, with probability $p(1-P_\infty)$. The sum of all these probabilities equals unity. As the concentration p approaches the threshold p_c, the pair connectedness length ξ diverges, $\xi \sim (p-p_c)^{-\nu}$. To mimic the geometric texture of the percolating network just above the percolation threshold, it is necessary that a regular model is self-similar (fractal) on smaller length scales than the connectedness length but becomes homogeneous on large length scales.

Now we try to imitate bond percolation with the help of a regular construction. The regular model is constructed by the following bond deletions and bond removals. Bonds on the square-centred lattice (see figure 1(b)) are recursively deleted via two rules and removed by a rule. First, we apply the first rule of bond deletion and the rule of bond removal. Two construction stages of our regular model are shown in figures 6(a) and (b). Crosses and triangles denote respectively the bonds deleted at the first and second stages. The dual lattice is shown by the broken lines. It is self-dual. Figure 7 shows part of the self-dual lattices at the third stage ($N=3$). Full triangles

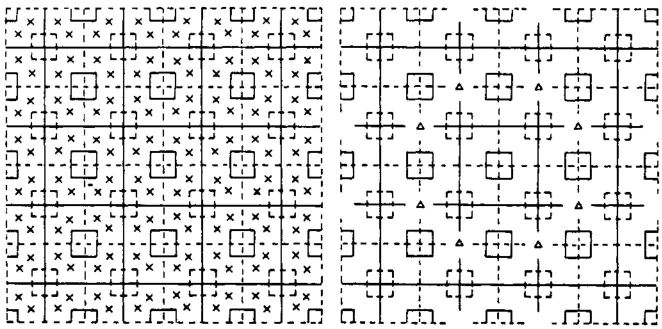

Figure 6. Two construction stages of the fractal model with the self-duality property. The lattices at the first and second stages are respectively indicated by (a) and (b). Crosses and triangles denote the bonds deleted at the first and second stages, respectively, according to the first rule of bond deletion.

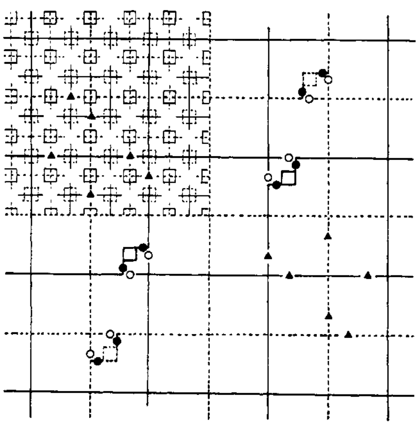

Figure 7. A part of the self-dual lattices at the third stage. Full triangles denote the bonds deleted at the third stage according to the first rule of bond deletion. The bonds marked by circles are removed to the nearest-neighbour positions indicated by the full circles and connected by the nearest neighbours according to the rule of bond removal. In the upper-left quarter in the figure, the lines with length scales smaller than the length of the nine units are omitted.

denote the bonds deleted at the third stage according to the first rule of bond deletion. The bonds marked by circles are removed to the nearest-neighbour positions indicated by the full circles and connected by the nearest neighbours according to the rule of bond removal. In the upper-left quarter in the figure, the lines with smaller length scales than the length of the nine units are omitted. Figure 8 represents part of the self-dual lattices obtained at the nth stage ($N > 3$). The bonds indicated by full triangles are deleted and the bonds by full circles are removed to the nearest neighbours. The lines are omitted with smaller length scales than the length of the 3^{N-1} units. The system obtained at the N stages appears to be a superlattice made by nodes separated by a distance of $\xi = 3^N$, connected by quasi-linear links. Within this model, the correlation between two sites at distance $r < \xi$ is via a single link, but this link is a branching curve. Schematic representation of the curve obtained by the bond deletions and removals is indicated by the lattice on the left-hand side in figure 9. In the limit N sufficiently large, the lattice approaches the Koch curve on the right-hand side. The differences (indicated by circles) of the left lattice from the right Koch curves disappear. The curve is identified as the branching Koch curve shown by figure 2(b). We obtain the square lattice with self-similar structures on smaller length scales than the connectedness length $\xi = 3^N$. The concentration $c_1(N)$ of bonds, deleted at the Nth stage via the first rule of bond deletion, is given by

$$c_1(N) = 6/9^N \qquad (N \geqslant 3) \tag{11}$$

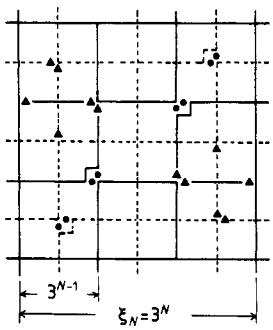

Figure 8. Bond deletions and removals at the Nth stage. The bonds indicated by full triangles are deleted and the bonds by full circles are removed to the nearest neighbours. The lines are omitted with smaller length scales than the length of the 3^{N-1} units.

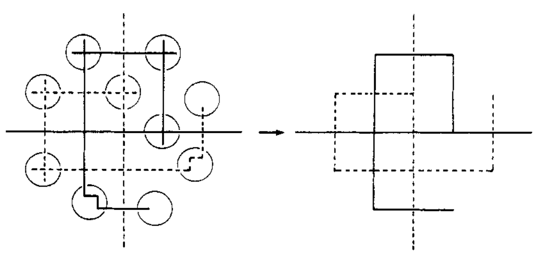

Figure 9. Schematic representation of the Koch curve obtained by bond deletions and removals is indicated by the lattices on the left-hand side. In the limit of N sufficiently large, the lattices approach the Koch curves on the right-hand side. The differences (indicated by circles) of the left lattices from the right Koch curves disappear.

where $c_1(1)=\frac{4}{9}$ and $c_2(2)=\frac{4}{81}$. We obtain the percolating network (infinite cluster) by use of the first rule of bond deletion and the rule of bond removal.

Secondly, we apply the second rule of bond deletion to the resultant lattice. The second rule is applied to the islands separated from the percolating network. This rule works at stages larger than $N=3$. Figures 10 and 11 show the constructions of finite clusters from an island separated from the percolating network at the fourth and fifth stages ($N=4$ and 5) respectively. The finite clusters are found to be a fractal with the initiator and the generator shown by figures 12(a) and (b). The number $n(N)$ of bonds per island, deleted at the Nth stage via the second rule of bond deletion, is

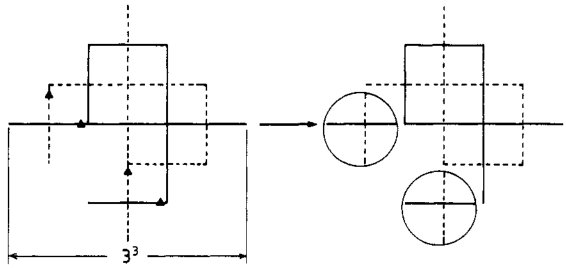

Figure 10. The construction of finite clusters from a island separated from the percolating network at the fourth stage ($N=4$). An island is shown on the left-hand side. Bonds into the island indicated by the full triangles are deleted by the second rule. The finite clusters on the right-hand side generate.

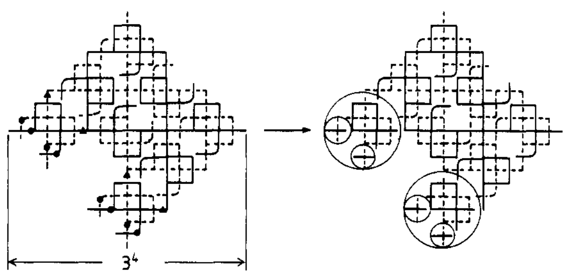

Figure 11. The construction of finite clusters from a island separated from the percolating network at the fifth stage ($N=5$). Bonds into the island indicated by the full triangles and circles are deleted by the second rule. The finite clusters on the right-hand side generate.

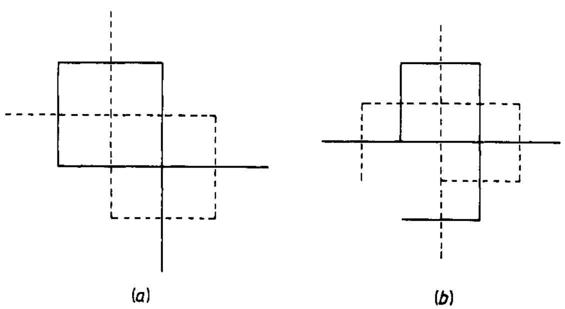

Figure 12. (a) The initiator and (b) the generator for finite clusters.

given by

$$n_2(N) = 4 + 4n_2(N-1) \qquad (N \geqslant 4) \tag{12}$$

where $n_2(3) = 0$. The concentration $c_2(N)$ of bonds is obtained by

$$c_2(N) = n_2(N)/9^N \qquad (N \geqslant 4). \tag{13}$$

The concentration $p(N)$ of bonds after N stages is given by

$$p(N) = 1 - \sum_{n=1}^{N} c_1(n) - \sum_{n=4}^{N} c_2(n). \tag{14}$$

When N is infinitely large, the concentration p approaches the critical value p_c:

$$p_c = \lim_{N\to\infty} p(N) = 1 - (40/81) - (158/19\,683)\left(\sum_{n=0}^{\infty} (1/9)^n\right) - (16/19\,683)\left(\sum_{n=0}^{\infty} (4/9)^n\right)$$
$$= 803/1620 (= 0.4956\ldots). \tag{15}$$

From (15) we obtain

$$\delta p (\equiv p(N) - p_c) \sim (\tfrac{4}{9})^N \sim \xi^{-2(1-\log 2/\log 3)}. \tag{16}$$

The connectedness length diverges as

$$\xi \sim (p - p_c)^{-\nu} \qquad \text{and} \qquad \nu = 0.5/(1 - \log 2/\log 3) (= 1.3547\ldots). \tag{17}$$

The value for the correlation length exponent agrees with that derived in a completely different fashion by Klein *et al* (1978) and was then thought to be perhaps exact. The most important feature of the regular model described above is that it is possible to get explicit expressions for the quantities characterising the approach towards the percolation threshold. The regular model is self-similar (fractal) on smaller length scales than the connectedness length, but becomes a homogeneous square lattice on large length scales. Our model is possessed of characteristic properties that the infinite cluster is composed of a backbone through which electrical current flows and dangling bonds hang on it and the backbone consists of multiply connected 'blobs' joined by singly connected 'links'. The self-similar structure of the regular model is constructed by hierarchical extrapolation. The generator of the fractal is given by the Koch curve shown in figure 2(*b*). The exponents, describing the power law dependence on scale length L of the Ohmic and Hall conductivities, are given by equations (7) and (8). By assuming the Einstein relation the spectral dimension d_s is given by $d_s = \log 64/\log 22$ $(= 1.3454\ldots)$ (Alexander and Orbach 1982, Alexander 1983).

The other important feature of the regular model is that it is possible to get explicit expressions for the quantities characterising the statistics of clusters defined in percolation. In order to obtain the cluster size distribution, one should note that the largest clusters generated in the kth stage of the process of bond deletions contain $s(k) \sim (8)^k$ bonds. For $s < (3^D)^N$ the cluster size distribution consists of a sum of delta functions:

$$n_s \sim \sum_{k=1}^{N} (1/9)^k \left(\sum_{n=0}^{N-k} (2/9)^n\right) \delta(s - (3^D)^k)\theta(1 - s/(3^D)^N). \tag{18}$$

By spreading the delta functions over the interval we obtain

$$n_s \sim (1/9)^k (1/3^D)^k \theta(1 - s/(3^D)^N). \tag{19}$$

Taking into account that $s \sim (3^D)^k$, one can arrive at the scaling form

$$n_s \sim s^{-\tau'}\theta(1 - s/s_\xi) \tag{20}$$

with $\tau' = 1 + \log 9/\log 8 = 1 + d/D$ where $s_\xi \sim (3^D)^N \sim (p-p_c)^{-\nu D}$, so $1/\sigma = \nu D$ is obtained. The scaling law $\beta/\nu = d - D$ is then satisfied. Our critical exponents are very close to the exact ones.

5. Summary

We present the fractal models with the self-dual property for the low-field Hall effect near percolation threshold. Two-dimensional percolation can be imitated by the regular model. The regular construction of percolation, to simulate the scaling properties in the two-dimensional bond percolation, is shown to be possessed of characteristic properties of the infinite and finite clusters. It is found that the model shows the typical percolation behaviour as a function of a parameter p. Table 1 lists the geometric and physical properties determined analytically by our fractal model. In table 1, the second line shows estimated scaling exponents for the two-dimensional (random) percolation to compare more completely our results with those of random percolation. Our critical exponents are very close to the exact ones.

Table 1. List of the physical and geometric properties of our regular model compared with other sources: [a]Stauffer (1985); [b]Kapitulnik and Deutscher (1984); [c]Herrmann and Stanley (1984); [d]Hermann *et al* (1984); [e]Lobb and Frank (1984); [f]Bergman (1983).

p_c	ν	τ'	D	D_b	t/ν	τ/ν
803/1620	$0.5/(1-\log_3 2)$	$1+\log_8 9$	$\log_3 8$	$\log_3 6$	$\log_3 \frac{11}{3}$	$2t/\nu$
(0.4956)	(1.354)	(2.056)	(1.892)	(1.630)	(0.9207)	
0.5[a]	1.33[a]	2.05[a]	1.90[b]	1.62[c]	0.97[d,e]	$2t/\nu$[f]

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